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On Artificial Viscosity*

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It is shown that the usual analysis for the implicit artificial viscosity of finite difference analogs of the linear advection equation is ambiguous, with different results obtained for transient and steady-state problems. The ambiguity is easily resolved for the inviscid equation, but for the advection-diffusion equation, the steady-state analysis is shown to be applicable to steady-state problems. It is demonstrated that the currently most popular methods, touted as having no artificial viscosity, actually do have such when applied to steady-state problems.

INTRODUCTION

"Artificial viscosity" is a particular kind of truncation error exhibited by some finite difference analogs of advection equations. The first use of the term was by von Neumann and Richtmyer [1], who explicitly added a viscosity-like term to the inviscid gas dynamic equations in order to allow the calculation of shock waves by what is now known as the "shock-smearing" or "through" method. Their explicit artificial viscosity term was deliberately made proportional to Δx^2, so as to assure mathematical consistency; that is, their *explicit* artificial viscosity term was indeed a second-order *truncation error.*

It has since been recognized that the same kind of artificially viscous behavior can be obtained, often inadvertently, just due to the truncation error of the FDE (finite difference equation). Noh and Protter [2] first presented an analysis of the *implicit* artificial viscosity of the upwind differencing method applied to the linear model advection equation

$$\zeta_t = -u\zeta_x . \tag{1}$$

For $u > 0$, the upwind differencing method for (1) gives the following FDE:

$$(\zeta_i^{n+1} - \zeta_i^n)/\Delta t = -u[(\zeta_i^n - \zeta_{i-1}^n)/\Delta x]. \tag{2}$$

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The truncation error is $O(\Delta t, \Delta x)$. Rewriting (2) in terms of the Courant number $c = u\Delta t/\Delta x$ gives, for $u =$ constant,

$$\zeta_i^{n+1} = \zeta_i^n - c(\zeta_i^n - \zeta_{i-1}^n). \tag{3}$$

For $c = 1$, the method gives $\zeta_i^{n+1} = \zeta_{i-1}^n$, which is the exact solution. The condition $c = 1$ is also the stability limit. For $c < 1$, the method introduces an artificial damping, in that the von Neumann stability analysis shows that the amplification matrix has eigenvalues $|\lambda| < 1$. Any method which has $|\lambda| < 1$ introduces such an artificial "damping," but a Taylor series expansion, as in the application of Hirt's stability analysis [3], shows that Eq. (3) is equivalent to

$$\zeta_t = -u\zeta_x + (u\Delta x/2)\,\zeta_{xx} - \tfrac{1}{2}\Delta t\zeta_{tt} + O(\Delta x^2, \Delta t^2). \tag{4}$$

The ζ_{tt} term in (4) is customarily evaluated from (1) for constant u as

$$\zeta_{tt} = -u\zeta_{xt} = -u(\zeta_t)_x = u^2\zeta_{xx}. \tag{5}$$

Using (5) in (4) gives

$$\zeta_t = -u\zeta_x + \alpha_e\zeta_{xx} + O(\Delta x^2, \Delta t^2), \tag{6}$$

where

$$\alpha_e = (u\Delta x/2) - (u^2\Delta t/2) = \tfrac{1}{2}u\Delta x(1 - c). \tag{7}$$

Since the method has introduced a nonphysical coefficient α_e of $\partial^2\zeta/\partial x^2$, we are justified in referring not only to the artificial damping, but more specifically, to artificial or numerical diffusion or numerical *viscosity* of the method. (Hirt [3] successfully uses $\alpha_e > 0$ as a necessary stability criterion.) For $c = 1$, (7) indicates $\alpha_e = 0$, a result consistent with the fact that the exact solution is obtained for $c = 1$.

Transient vs Steady-State Analyses

The above analysis has been used by many authors to describe the artificial viscosity of various methods, and the results are widely accepted as being applicable to multidimensional problems, with and without physical viscous terms. But the interpretation of α_e in multidimensional, viscous and/or steady-state problems is not as straightforward as it might appear. Suspicion arises when one considers the form of (7) which shows an α_e dependent on Δt through the Courant number c. Consider a problem in which a steady state has developed, with $\zeta_i^{n+1} = \zeta_i^n$. Once this condition is reached[1], both the FDE (2) and computational experience with the upwind differencing method in multidimensional problems indicate that a change in Δt does not change the steady-state solution. Yet (7) would indicate that a reduction in Δt increases α_e (through c). If the concept of artificial viscosity α_e means anything, it would appear that the FDE solution should depend on α_e; but we see that we can change α_e through Δt, and not change the steady-state solution.

Alternate to the above analysis of the transient equation, one can instead analyze for the α_e effect after assuming that a steady state exists. Setting $\zeta_i^{n+1} = \zeta_i^n$ in (2) and expanding in a Taylor series, we obtain a steady-state α_e, denoted by α_{es}, as

$$\alpha_{es} = \tfrac{1}{2} u \Delta x. \tag{8}$$

In this formulation, $\alpha_{es} \neq f(\Delta t)$ and the steady-state independence of Δt is not suspect.

The resolution of the ambiguity between the two different expressions (7) for α_e and (8) for α_{es} is readily accomplished by recognizing that, for the inviscid model equation, the only possible steady-state solution with u = constant is the trivial solution $\zeta_i^n = \zeta_1^n$ = constant. In this case, $\partial^2\zeta/\partial x^2 = 0$, permitting an arbitrary form for α_e. The question is, which analysis (if either) is appropriate to problems with (a) diffusion terms present, (b) dimensions greater than one, (c) spatially varying or nonlinear advection velocities u?

The question may be easily and unambiguously answered for the addition of diffusion terms to (1), with a physical diffusion coefficient α

$$\zeta_t = -u\zeta_x + \alpha\zeta_{xx}. \tag{9}$$

Using upwind differencing on the advection term and forward-time centered-space differencing on the diffusion term gives

$$\zeta_i^{n+1} = \zeta_i^n - c(\zeta_i^n - \zeta_{i-1}^n) + d(\zeta_{i+1}^n - 2\zeta_i^n + \zeta_{i-1}^n), \tag{10}$$

where $d = \alpha\Delta t/\Delta x^2$. The steady-state analysis for Eq. (9) gives

$$0 = -u\zeta_x + (\alpha + \alpha_{es})\,\zeta_{xx} + O(\Delta x^2), \tag{11}$$

where α_{es} is again given by the steady-state form (8). The transient analysis is altered, because (5) must be replaced by

$$\zeta_{tt} = (-u\zeta_x + \alpha\zeta_{xx})_t = u^2\zeta_{xx} - 2u\alpha\zeta_{xxx} + \alpha^2\zeta_{xxxx} \tag{12}$$

and (6) must be replaced by

$$\zeta_t = -u\zeta_x + (\alpha + \alpha_e)\,\zeta_{xx} + O(\Delta x^2, \Delta t^2) + \text{HOD}, \tag{13}$$

[1] We do not wish to confuse the matter by considering iteration convergence criteria at this point.

where the higher-order derivative terms are

$$\mathrm{HOD} = \Delta t[u\alpha\zeta_{xxx} - (\alpha^2/2)\,\zeta_{xxxx}] \tag{14}$$

and α_e is again the transient form given by (7). Hirt [3] ignores the HOD in (14) and in this way successfully predicts the transient stability behavior, but we are interested in the α_e appropriate for a steady-state solution, and we must retain the HOD.

For any steady-state solution, (9) gives

$$\zeta_{xxxx} = (u/\alpha)\,\zeta_{xxx} = (u/\alpha)^2\,\zeta_{xx} = (u/\alpha)^3\,\zeta_x\,. \tag{15}$$

We now apply these relations (15) for a steady state to the result of the transient analysis. Assuming a steady state in (13), using (14) and (15), and substituting (7) for α_e gives

$$\begin{aligned} 0 = &- u\zeta_x + \alpha\zeta_{xx} + (u\Delta x/2)\,\zeta_{xx} - (u^2\Delta t/2)\,\zeta_{xx} + \Delta t u\alpha(u/\alpha)\,\zeta_{xx} \\ &- \Delta t(\alpha^2/2)(u/\alpha)^2\,\zeta_{xx} + O(\Delta x^2, \Delta t^2), \end{aligned} \tag{16}$$

$$0 = -u\zeta_x + (\alpha + \alpha_{es})\,\zeta_{xx} + O(\Delta x^2, \Delta t^2), \tag{17}$$

where the steady state α_{es} is given by (8). It is thus clear that although the transient α_e of (7) may be appropriate for Hirt's stability analysis, the steady-state form α_{es} of (8) is appropriate when a steady-state condition has been reached, even though the transient equation is analyzed.

It may be argued that the last relation of (15) could be used to eliminate $\alpha_{es}\zeta_{xx}$ from (11), thus leading to the conclusion that no artificial viscosity coefficient α_{es} is present, but rather that an "artificial advection velocity" u_{es} is present, as in

$$0 = -(u - u_{es})\,\zeta_x + \alpha\zeta_{xx} + O(\Delta x^2), \tag{18}$$

where

$$u_{es} = \alpha_{es}(u/\alpha) = \tfrac{1}{2}u^2\Delta x/\alpha. \tag{19}$$

However, the "artificial velocity" term in (18) must still be interpreted as producing an artificial viscous effect, even though the $\alpha_{es}\zeta_{xx}$ term has been removed. The steady-state solution is not determined by α and u independently, but only by their ratio u/α, along with the boundary conditions. When the proper length normalizing of the spatial domain of definition is taken into account, this ratio u/α is a Reynolds number. An artificial viscous effect is then simply any effect which reduces the effective Reynolds number u/α. In (11), the artificial viscous effect is expressed as an artificial increase in α, which reduces u/α to $u/(\alpha + \alpha_{es})$. In (18), the artificial viscous effect is expressed as an artificial decrease in u, which reduces u/α to

$(u - u_{es})/\alpha$. Thus, both α_{es} in (11) and u_{es} in (18) act to reduce the effective Reynolds number and therefore have an artificial viscous effect.

There is, in fact, a quantitative ambiguity in these two steady-state anaylses, due to the use of (15) in the finite difference solution, whereas (15) is only applicable to the continuum solution. Equation (11) has a factor

$$\frac{u}{\alpha + \alpha_{es}} = \frac{u}{\alpha}\left(\frac{1}{1 + \frac{1}{2}u\Delta x/\alpha}\right), \tag{20}$$

whereas (18) has a factor

$$(u - u_{es})/\alpha = (u/\alpha)(1 - \tfrac{1}{2}u\Delta x/\alpha). \tag{21}$$

But since $1/(1 + \epsilon) = 1 - \epsilon + O(\epsilon^2)$, these two equations (20) and (21) for the artificial viscous effect are equal, to within a truncation error term of order Δx^2, provided that

$$\tfrac{1}{2}u\Delta x/\alpha \ll 1. \tag{22}$$

This is obviously true as $\Delta x \to 0$, in which case (15) becomes applicable to the FDE. [Equation (22) is the familiar requirement for formal accuracy of the upwind difference method, that the computational cell Reynolds number $u\Delta x/\alpha$ be $\ll 2$.]

Similarly, (15) might be used in (11) to express the first-order truncation error as a coefficient of ζ_{xxx} just as legitimately. But since we have no such term in the original continuum equation, this exercise does not lend itself to a fruitful interpretation of the physically analogous behavior of the FDE.

We also remark that if problems are considered with boundary conditions either of the form

$$\zeta(0) = a, \qquad \zeta_x(1) = b \tag{23}$$

or

$$\zeta(0) = a, \qquad \zeta(1) = b, \tag{24}$$

the resulting steady-state solution is

$$\zeta(x) = C_1 + C_2 e^{xu/\alpha} \tag{25}$$

with $C_2 \neq 0$. This solution gives nonzero values for all spatial derivatives. Unlike the situation for the inviscid equation, the distinction between the α_e of (7) and the α_{es} of (8) is then important.

For multidimensional problems with nonlinear coefficients, the resolution of the transient and steady-state analyses is not so neat. Both analyses give different values of α_e or α_{es} in different directions, each of the form (7) or (8). But the transient form α_e in (7) depends on (5), an equation which is not applicable to multidimensional and/or nonlinear problems. Further, the multidimensional transient analysis predicts that the steady-state solution for the upwind differencing method is a function of (Δt), which disagrees with computational experience. Thus, the steady-state analysis does appear to be appropriate for multidimensional nonlinear steady-state problems.

Analyses of Other Methods

In Table I, we present the results of both the transient and steady-state analyses for the artificial viscosity of various methods, based on the inviscid model equation (1). (Higher-order terms in the transient expansion have been given by Tyler [4].) The steady-state results for the inviscid equation are identical to those results obtained from the viscous equation (9), using for the viscous term any of the usual methods based on second-order space-centered differences; these include FTCS, fully implicit, ADI, Cheng–Allen, Crocco, Saul'yev, Adams–Bashforth methods [6], etc. For u = constant, the upwind difference method is equivalent to the "donor cell" [5] or "second upwind difference" method [6], which uses cell-averaged advection velocities at cell interfaces. It has nonzero artificial viscosity in both analyses, for $c < 1$. The forward-time, centered-space (FTCS) method is, of course, unstable in the absence of physical viscous terms, and accordingly has $\alpha_e < 0$ in the transient analysis [3]. The Lax method [7] is still frequently used, and also has nonzero artificial viscosity in both analyses, for $c < 1$.

Leith's method [8] (see also [2]) is very important. It is based on a second-order Taylor series expansion of (1) in time. For the model equation (1), Leith's method is algebraically identical to other methods based on the second-order time expansion, such as the Lax–Wendroff method [9], the Richtmyer [10] and other two-step Lax–Wendroff methods, Moretti's method [11], and MacCormack's method [12]. Leith's method also is involved in Fromm's method of zero average phase error [13], and even is related to Rusanov's method [14] for certain combinations of parameters. Significantly, $\alpha_e = 0$ is indicated only in the transient analysis.[2] From the steady-state analysis, $\alpha_{es} = \frac{1}{2}u^2 \Delta t$ is indicated, implying that $\alpha_{es} = 0$ only as $\Delta t \to 0$. There is no danger of misinterpretation of higher-order terms here, because the method is *algebraically* equivalent to the FTCS method applied to the advection-diffusion equation (9) with the physical $\alpha = \frac{1}{2}u^2 \Delta t$. Unlike the example of upwind differencing considered earlier, the FDE and computational experience now indicate that the steady-state solution will depend on Δt.

[2] Leith [8] was only concerned with the transient problem, of course. The present work is not to be construed as a criticism of Leith's work.

TABLE I

Implicit artificial viscosities from transient and steady-state analyses for various finite difference methods applied to $\zeta_t = -u\zeta_x$, with $c = u\Delta t/\Delta x$.

Description	Method	Transient	Steady	Formal truncation order
1. Upwind	$\zeta_i^{n+1} = \zeta_i^n - c(\zeta_i^n - \zeta_{i-1}^n)$	$\alpha_e = (u\Delta x/2)(1 - c)$	$\alpha_{es} = (u\Delta x/2)$	$O(\Delta t, \Delta x)$
2. FTCS	$\zeta_i^{n+1} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n)$	$-(u^2\Delta t/2)$	0	$O(\Delta t, \Delta x^2)$
3. Lax	$\zeta_i^{n+1} = (1/2)(\zeta_{i+1}^n + \zeta_{i-1}^n)$ $-(c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n)$	$(\Delta x^2/2\Delta t)(1 - c^2)$	$\Delta x^2/2\Delta t$	$O(\Delta t, \Delta x^2, \Delta x^2/\Delta t)$
4. Leith[a]	$\zeta_i^{n+1} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n)$ $+ (c^2/2)(\zeta_{i+1}^n - 2\zeta_i^n + \zeta_{i-1}^n)$	0	$u^2\Delta t/2$	$O(\Delta t^2, \Delta x^2)$
5. Matsuno[b]	$\overline{\zeta_i^{n+1}} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n)$ $\zeta_i^{n+1} = \zeta_i^n - (c/2)(\overline{\zeta_{i+1}^{n+1}} - \overline{\zeta_{i-1}^{n+1}})$	$u^2\Delta t$	0	$O(\Delta t, \Delta x^2)$

[a] Also Lax–Wendroff, 2-step Lax–Wendroff, Moretti, MacCormack.

[b] Also Brailovskaya, Cheng–Allen.

The two-step Matsuno method [15] of differencing the advection terms has also been used for compressible flow by Brailovskaya [16], using the same approach on the viscous terms, and by Allen and Cheng [17], using a special treatment of the physical diffusion terms which successfully removes the additional Δt restriction present in Brailovskaya's method due to the diffusion term. The Matsuno method requires special mention, because of a further ambiguity in the steady-state α_{es} analysis. This two-step method for (1) is written as

$$\overline{\zeta_i^{n+1}} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n), \tag{26a}$$

$$\zeta_i^{n+1} = \zeta_i^n - (c/2)(\overline{\zeta_{i+1}^{n+1}} - \overline{\zeta_{i-1}^{n+1}}). \tag{26b}$$

The $\overline{(n+1)}$ values are provisional or intermediate values. The method may be interpreted as a first iterative approximation to the fully implicit method. For the purposes of stability analysis and artificial viscosity analysis, (26) may be rewritten as a single equation

$$\zeta_i^{n+1} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n) + (c^2/4)(\zeta_{i+2}^n - 2\zeta_i^n + \zeta_{i-2}^n). \tag{27}$$

The equivalence of (27) to the two-step method (26) holds only for the model equation (1) at interior points; the presence of boundaries and nonlinearities destroys this equivalence. The last term of (27) is recognized as the usual 3-point expression for $\alpha\partial^2\zeta/\partial x^2$, but written over a mesh spacing of $2\Delta x$ rather than Δx. With this interpretation, the steady-state analysis would indicate $\alpha_{es} = 2u^2\Delta t$. However, the higher-order terms enter into the behavior of the equation in an unexpected and fortunate manner. Each of the two steps (26a) and (26b) has the same operator form, i.e.,

$$\overline{\zeta^{n+1}} = \zeta^n + L(\zeta^n), \tag{28a}$$

$$\zeta^{n+1} = \zeta^n + L(\overline{\zeta^{n+1}}). \tag{28b}$$

(This is in contrast to the two-step Lax–Wendroff methods, for example.) Allen and Cheng [17] noted the significant fact that, when a steady state is reached with this method, not only does $\zeta^{n+1} = \zeta^n$, but also $\overline{\zeta^{n+1}} = \zeta^n$. Using this information, the steady-state analysis for α_{es} can be applied to each step of (26) separately, rather than to (27). The result is $\alpha_{es} = 0$, as for the FTCS method. This conclusion has been verified in the present study by one-dimensional tests, which exhibit a steady-state solution which is not a function of Δt, in contrast to the analysis of (27) and in contrast to Leith's method.

A Two-Dimensional Experiment

To test the applicability of the results from the one-dimensional model equation (1) to the two-dimensional gas dynamic equations, a numerical experiment using Moretti's inviscid blunt body program [18] was run. A 6° half-angle sphere-cone was run at a free-stream Mach number $M = 10$, with an ideal gas and a ratio of specific heats $\gamma = 1.4$. The program utilizes shock patching in a curvilinear mesh system which adjusts as the solution develops. Since the shock is correctly maintained as a discontinuity in this program, the present results are not confused by the postshock oscillations of the "through" or shock-smearing calculation methods. An extremely coarse mesh was chosen to exaggerate the α_{es} effects; the mesh had only three mesh points (two intervals) between the body and the shock, and only five mesh points along the body. The object of the experiment was to show that the steady-state solution obtained with Moretti's method is a function of the Δt used, as indicated by the steady-state analysis for α_e. (This behavior is in contrast to that of the upwind difference method considered earlier, and indeed to most other finite difference methods.)

The most sensitive location was found to be the (2, 3) point, in the center of the mesh. The Δt was changed by the program input parameter STAB; for STAB $= 1$, the Δt used was about 0.94 of the linear stability limit for a square mesh. The first segment of solution A, shown in Fig. 1, was run out to 3000 time steps with STAB $= 1$, giving a dimensionless time $T = 15.82$. This represents a rather unequivocal steady state, with the normalized density ρ changing by only 2.5×10^{-6} in the last 200 time steps, or less than $2.74 \times 10^{-7}\,\%$ per time plane. Then the second segment of solution A was obtained by changing the critical time-step

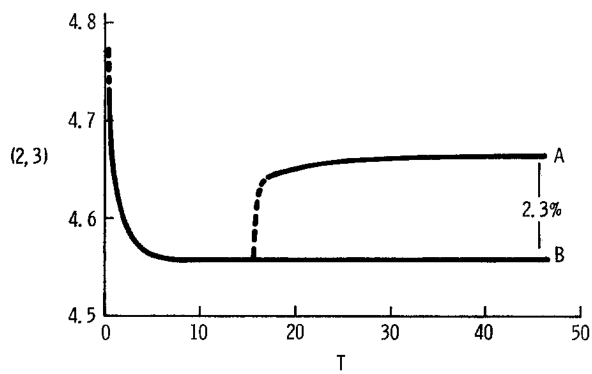


Fig. 1. Late time density solutions at point (2, 3) using Moretti's method. 6° sphere-cone, $M = 10$, $\gamma = 1.4$, 3×5 mesh. For $T < 15.82$, $\Delta t_A = \Delta t_B \simeq 0.94\,\Delta t$ crit. For $T \geqslant 15.82$, $\Delta t_A \simeq 1/5\,\Delta t_B$.

multiplier STAB to 1/5. Nothing else was altered. This computation was continued for an additional 28,000 time steps, at which $T = 46.45$. This gave a new steady-state solution, with ρ changing by only 4.32×10^{-4} in the last 1000 time planes. As a further check, a second solution B was run using the larger Δt (STAB = 1) all the way out to $T > 46.45$.

The difference between the two steady-state solutions at $T = 46.45$ is shown in Fig. 1, and is presented tabularly in Table II. At the most sensitive point (2, 3), the normalized densities differed by 2.3 %, the normalized pressures by 3 %, and the normalized shock stand-off distance by 0.6 %.

TABLE II

Steady-state solutions at point (2, 3) using Moretti's method. 6° sphere-cone, $M = 10$, $\gamma = 1.4$, 3×5 mesh, ρ, P, r_s = normalized values of density, pressure, and shock stand-off distance. $\Delta t_A \simeq 1/5\, \Delta t_B$.

	$\rho(2, 3)$	$P(2, 3)$	r_s
Solution A	4.664	76.54	1.142
Solution B	4.559	74.25	1.149
Percent difference	2.3 %	3.0 %	0.6 %

That the percentage difference between the two different solutions is small is to be expected, since the blunt body problem is known from physical and numerical experiments to be quite insensitive to Reynolds number. The FDE solution is then only a weak function of α_{es} and Δt, especially since inviscid boundary conditions are used on the surface so that no boundary layer develops, and since the shock is treated as a discontinuity. The numerical solutions obtained by this and other methods, using both implicit and explicit artificial viscosities, are certainly valid approximations. The significant point is that the two-dimensional steady-state solution obtained did depend on Δt, supporting the one-dimensional analysis for α_{es} which indicates that the method does exhibit an artificial viscosity effect in the steady state. A further indication of a viscosity effect was obtained from two solutions in a finer (5×7) mesh. The solution with STAB = 1 was steady to all four significant figures printed out for that test, whereas a "steady" solution obtained with STAB = 1/10 exhibited a persistent oscillation of ± 1 in the second significant figure of the density. This behavior is again consistent with the indication of the steady-state analysis for which $\alpha_{es} \propto \Delta t$.

Interpretation of the Lax–Wendroff Methods

The interpretation of the artificial viscosity for the Lax–Wendroff methods involves the resolution of paradoxical statements. On the one hand, we have the facts that (1) the transient analysis indicates $\alpha_e = 0$ and a formal truncation error for the inviscid equation of $O(\Delta x^2, \Delta t^2)$, and (2) the exact transient solution $\zeta_i^{n+1} = \zeta_{i-1}^n$ is obtained for $c = 1$ (and in $2D$ with time splitting, $\zeta_{ij}^{n+1} = \zeta_{i-1,j-1}^n$ for $c_x, c_y = 1$). On the other hand, we know that the steady-state analysis indicates $\alpha_{es} > 0$, with the Lax–Wendroff methods for the *inviscid* equation being *algebraically equivalent* to centered-space differencing of the steady *viscous* equation with $\alpha_{es} = u^2 \Delta t/2$.

The paradox is due to the effect of boundary conditions. In order to resolve this paradox, we consider the equation

$$-A\zeta_x + B\zeta_{xx} = 0, \tag{29a}$$

$$\zeta(0) = 0, \qquad \zeta(1) = 1. \tag{29b}$$

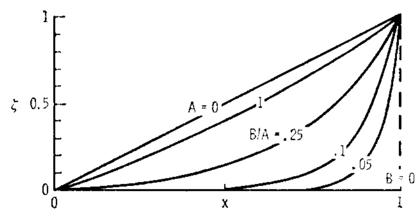
Using centered differences, we have

$$-(A/2\Delta x)(\zeta_{i+1} - \zeta_{i-1}) + (B/\Delta x^2)(\zeta_{i+1} - 2\zeta_i + \zeta_{i-1}). \tag{30}$$

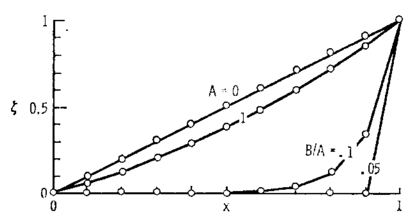
The solution to (29) is

$$\zeta(x) = (1 - e^{xA/B})/(1 - e^{A/B}). \tag{31}$$

This solution to the continuum equation is plotted in Fig. 2a for various parameters A and B. The corresponding finite difference solutions for $\Delta x = 1/10$ are plotted in Fig. 2b. In order to interpret α_{es}, we must examine these solutions from the viewpoints of both the viscous and the inviscid equations.



a. CONTINUUM SOLUTIONS



b. FINITE DIFFERENCE SOLUTIONS

FIG. 2 Continuum and finite difference solutions to $-A\zeta_x + B\zeta_{xx} = 0$, $\zeta(0) = 0$, $\zeta(1) = 1$. Centered differences, $\Delta x = 1/10$.

We first consider the viscous equation. For $A/B = 0$, the first derivative (advection) drops out; we have the simple straight-line continuum solution $\zeta(x) = x$, and the exact finite difference solution $\zeta_i = (i - 1)\,\Delta x$. For $A/B > 0$, the ζ profile is blown downstream. As A/B becomes large, the continuum solution becomes $\zeta(x) \simeq 0$ up to the neighborhood of $x = 1$, where a rapid increase in ζ is required in order to meet the second boundary condition of $\zeta(1) = 1$. For $B = 0$, the continuum solution becomes $\zeta(0) = 0$ everywhere; the second boundary condition $\zeta(1) = 1$ cannot be met, and is extraneous since it would overspecify the problem. As this condition $B = 0$ is approached in the limit, we have the classical singular perturbation problem in the small parameter B/A, in which the order of the differential equation is reduced as $B/A \to 0$. For the finite difference equation, the behavior analogous to the singular perturbation problem occurs at $A/B = 20$ (more generally, $A\Delta x/B = 2$). At this condition, the FDE solution is $\zeta_i = 0$ for $i \leqslant 10$, and $\zeta_{11} = 1$. This FDE solution may be interpreted here as a qualitatively correct *viscous* behavior. But for $A\Delta x/B > 2$, oscillations and undershoot ($\zeta_{10} < 0$) develop as described in [6]. In terms of the *viscous* steady-state equation, this dividing condition $A\Delta x/B = 2$ corresponds to a cell Reynolds number $R_c \equiv u\Delta x/\alpha = 2$.

We next consider the inviscid equation using a Lax–Wendroff method. The condition $A\Delta x/B = 2$ now corresponds to $c = 1$. The exact transient solution is obtained as $\zeta_i^{n+1} = \zeta_{i-1}^n$; for $n > 10$, this gives the exact steady-state solution of $\zeta_i = \zeta_1 = 0$ for $i \leqslant 10$. For this condition of $c = 1$, the extraneous boundary value $\zeta_{11} = 1$ does not feed forward and influence the solution at interior points. The extraneous value $\zeta_{11} = 1$ is an error in this inviscid interpretation, but it is a purely *local* error for $c = 1$. Thus, there exists no contradiction with the formal truncation error of the method which implies an error of $O(\Delta x^2)$ for the steady-state problem. (This localness of the outflow error likewise removes the ambiguity for the upwind difference method and others which give the exact solution for $c = 1$.) The point is that the FDE solution for $A\Delta x/B$ can be validly interpreted as *either* a qualitatively correct viscous solution with $R_c = 2$, which solution includes $\zeta_{11} = 1$, or as an exactly correct inviscid solution at $c = 1$, which solution does not include the extraneous local boundary error $\zeta_{11} = 1$.

However, this interpretation is altered by two situations of practical importance; nonunity Courant numbers, and the addition of viscous terms to the equations.

The first situation which alters the usual analysis is the case of nonunity Courant number. For $c < 1$, the Lax–Wendroff methods no longer give the exact solution at interior points, but the formal truncation error is still $O(\Delta x^2, \Delta t^2)$. However, the FDE solution with a fixed outflow boundary value can only be interpreted as a viscous solution with $\alpha_{es} = u^2\Delta t/2$, and with only first order accuracy. The $O(\Delta x)$ error has been introduced by the requirement for the extraneous outflow boundary condition, which feeds forward for $c < 1$ and produces the artificially viscous behavior. Since the usual analysis for formal truncation error does not include boundary effects, it is inadequate in the present case of $c < 1$, and the true first order accuracy of the steady-state FDE solution is indicated by the steady-state analysis for artificial viscosity.

We have considered only a fixed outflow boundary condition, presumed to be in error. It is possible that this outflow error could actually be ordered (i.e., $\sim\Delta x^p$) using any of several methods [6]. The outflow error for $c < 1$ will almost certainly be $O(\Delta x)$, and the remarks made elsewhere in this paper are based on that assumption. But in the event that the ouflow error were only $O(\Delta x^2)$ then the Lax–Wendroff solution for $c < 1$ would be $O(\Delta x^2)$; however, the solution would still be artificially viscous in the sense that there exists a nonzero coefficient of ζ_{xx}.

The second situation which alters the usual analysis is the addition of viscous terms in the continuum equation. The viscous terms cannot be treated by the Lax–Wendroff time differencing, which would be unstable, but several authors have used FTCS differencing for the viscous terms, as in

$$\zeta_i^{n+1} = \zeta_i^n - (c/2)(\zeta_{i+1}^n - \zeta_{i-1}^n) + (c^2/2)(\zeta_{i+1}^n - 2\zeta_i^n + \zeta_{i-1}^n) + d(\zeta_{i+1}^n - 2\zeta_i^n + \zeta_{i-1}^n), \quad (32)$$

where $d = \alpha\Delta t/\Delta x^2$, as before. (In the two-step Lax–Wendroff methods, the viscous term usually has been added in the second step only.) Here, the steady-state solution is clearly a viscous one, where the viscous term is the sum of the intended (physical) α and the artificial α_{es}. There is no contradiction of the usual truncation error analysis here, because the $O(\Delta x^2, \Delta t^2)$ result is not obtained from (32), which is readily seen to be only first order accurate. Note also that, with the physical viscous term present, large Courant numbers actually aggravate the α_{es} error. From Table I it is easily shown that the α_{es} of the Lax, Lax–Wendroff and upwind differencing methods are in the ratios $1/c$:1:c, respectively. For regions near the stability limit $c = 1$, the artificial viscosity of a Lax–Wendroff–FTCS method is virtually the same as the upwind difference method; within a boundary layer, $c < 1$, and the Lax–Wendroff–FTCS method will be more accurate, as in Ref. [12].

To summarize, the Lax–Wendroff methods do give the exact solution of the model equation, in both transient and steady-state cases, for $c = 1$ and no viscous terms present. The boundary error at outflow is purely local. The usual truncation error analysis is applicable, and indicates errors of only $O(\Delta x^2, \Delta t^2)$ with no artificial viscosity effect. The steady-state analysis showing $\alpha_{es} > 0$ is inappropriate. But for $c < 1$ in the inviscid equation, any outflow boundary error does have global effects which invalidate the usual truncation error analysis of the interior point equations and which introduce an artificially viscous behavior. Also, for the addition of viscous terms, the usual truncation error analysis is not applicable, and the method has an artificial viscosity effect which is aggravated by near-unity

Courant numbers. In both these cases of $c < 1$ and/or additional viscous terms, the steady-state analysis for artificial viscosity is appropriate, and shows $\alpha_{es} = u^2 \Delta t/2$ and first-order formal accuracy.

Implications to Other Methods

Although we have not tested the following methods experimentally, the implications of the analyses on the model equation (1) are as follows (see Ref. [6] for references and details). The midpoint leapfrog, the Crocco, Adams–Bashforth, Heun, fully implicit, Crank–Nicholson, and the various ADI methods would have zero artificial viscosity in the steady state, except when upwind differencing is used for the advection terms as has been done in some ADI solutions. The multistep Strang, Abarbanel and Zwas, Fromm [13], and the Crowley methods would have a persistent nonzero α_{es} in the steady state. The only known methods for which the analyses indicate zero artificial viscosity in both the transient and steady-state analyses are the midpoint leapfrog method, the Arakawa method, the angled derivative method of Roberts and Weiss [19], and those ADI methods which have a truncation error of $O(\Delta x^2, \Delta t^2)$. Each of these has other disadvantages, of course.

It is interesting to note that the expression for α_{es} of the Leith method, $\alpha_{es} = \frac{1}{2}u^2 \Delta t$, does not contain Δx directly. Thus, as $\Delta x \to 0$, the $\alpha_{es} \to 0$ only because of the Courant number restriction on stability, which requires $\Delta t \to 0$ as $\Delta x \to 0$. If a method were devised which used the second-order time expansion of the Leith (Lax–Wendroff, etc.) method but which was unconditionally stable, the α_{es} effect would persist even as $\Delta x \to 0$, for fixed Δt.

Final Remarks

We have four final remarks on the interpretation of artificial viscosity:

(1) The truncation error analysis indicates the *order* of the error, which is strictly applicable only as $\Delta x, \Delta t \to 0$. In a practical computation, we are generally interested not in the order of the truncation error, but in the *size* of the truncation error, for some Δx and Δt [6]. Thus, the addition of some miniscule viscous term (say $\text{Re} = 10^6$ for $\Delta x = 1/10$) *formally* deteriorates the truncation error of a Lax–Wendroff–FTCS calculation to $O(\Delta x)$, but the *size* of the error remains entirely negligible for $c = 1$. Note also that the *size* of the truncation error varies smoothly for $c \leqslant 1$, although the *order* jumps discontinuously (singularly) from the exact solution at $c = 1$ to $O(\Delta x)$ for $c < 1$.

(2) For multidimensional problems, the most important effect of viscosity, in the sense of producing a difference between viscous and inviscid solutions, is

usually not so much in the appearance of viscous terms at interior points, but in the enforcement of no-slip boundary conditions. Thus, Kentzner [20] has indicated that fairly accurate approximations to inviscid solutions can be obtained with Re as low as 300 in a reasonable mesh, provided that the inviscid (slip) boundary conditions are used. This means that inviscid solutions can be accurate even though artificial viscosity is present; however, the error may be somewhat more significant for viscous problems. (In assessing the α_{es} error of FDE solutions for drag coefficient C_D, for example, it is important to look not for some small error in C_D, but for a shift in Re to get the same C_D. This is obviously appropriate because of the usual weak sensitivity of flows to Re.)

(3) In multidimensional problems, the α_{es} terms depend on u and v, which are defined with respect to the Eulerian mesh. This means that different spatially varying α_{es} apply in the x and y directions, and tend to zero near stationary no-slip walls. Thus, "equivalent Re" interpretations are not possible even for viscous solutions, except in a qualitative sense, and viscous FDE solutions with nonzero α_{es} are often more accurate than might be expected from evaluating α_{es} based on freestream conditions. However, such solutions are not Galilean invariant [5]. Also, solutions for rotating bodies might exhibit anamolous behavior due to different α_{es} on the advancing and the retreating sides.

(4) Several methods are available [6] for freeing the outflow computational boundary condition in multidimensional flows. These will tend to reduce the upstream error associated with $c < 1$ in the inviscid equations.

Summary

It has been demonstrated that the usual method of analysis for the artificial viscosity of finite difference analogs for the advection terms is ambiguous, with different results being obtained for the transient and the steady-state analyses. The analysis indicates that many methods which have been touted as having no artificial viscosity, notably the Leith, Lax–Wendroff, two-step Lax–Wendroff, Moretti, and MacCormack methods, do have a Δt-dependent artificial viscosity effect when a steady-state solution is obtained for viscous flow and/or for $c < 1$. Viscous steady-state solutions obtained using these methods with Courant-numbers $c \simeq 1$ have only first order formal accuracy.

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References

1. J. von Neumann and R. D. Richtmyer, A method for the numerical calculations of hydro dynamical shocks, *J. Appl. Phys.* **21** (1950), 232.
2. W. F. Noh and M. H. Protter, *J. Math. & Mech.* **12** (1963), 149.
3. C. W. Hirt, *J. Comput. Phys.* **2** (1968), 339.
4. L. D. Tyler, "Proc. Second International Conference on Numerical Fluid Dynamics," (M. Holt, Ed.), Springer-Verlag, Berlin, 1971.
5. R. A. Gentry, R. E. Martin, and B. J. Daly, *J. Comput. Phys.* **1** (1966), 87.
6. P. J. Roache, "Computational Fluid Dynamics," to be published.
7. P. D. Lax, *Commun. Pure Appl. Math.* **7** (1954), 159.
8. C. E. Leith, "Methods in Computational Physics" (B. Alder, Ed.), Vol. 4, p. 1, 1965.
9. P. Lax and B. Wendroff, *Commun. Pure Appl. Math.* **13** (1960), 217.
10. R. D. Richtmyer, "A Survey of Difference Methods for Non-Steady Fluid Dynamics," NCAR Tech. Note 63–2, 1963.
11. G. Moretti and M. Abbett, *AIAA J.* **4** (1966), 2136.
12. R. W. MacCormack, AIAA Paper No. 69-354 (1969). Also, R. W. MacCormack, "Proc. Second International Conference on Numerical Fluid Dynamics" (M. Holt, Ed.), Springer-Verlag, Berlin, 1971.
13. J. E. Fromm, *J. Comput. Phys.* **3** (1968), 176.
14. V. V. Rusanov, *Zh. Vych. Mat. Mat. Fiz.* **1** (1961), 267. Nat. Res. Council of Canada TT-1027, 1962.
15. D. K. Lilly, *U. S. Monthly Weather Review* **93** (1965), 11.
16. I. Yu. Brailovskaya, *Sov. Phys.—Dokl.* **10** (1965), 107.
17. J. S. Allen and S. I. Cheng, *Phys. Fluids* **13** (1970), 37.
18. G. Moretti and G. Bleich, "Three-Dimensional Inviscid Flow About Supersonic Blunt Cones at Angle of Attack," SC-CR-68-3728, Sandia Laboratories, Albuquerque, New Mexico, September, 1968.
19. K. V. Roberts and N. O. Weiss, *Math. Comp.* **20** (1966), 271.
20. C. P. Kentzner, *J. Comput. Phys.* **6** (1970), 168.